Subdivisions of oriented cycles in digraphs with large chromatic number

Nathann Cohen, Frédéric Havet, William Lochet, Nicolas Nisse

Subdivisions of oriented cycles in digraphs with large chromatic number

Nathann Cohen [1], *Frédéric Havet* [2] William Lochet [2] and *Nicolas Nisse* [3]

[1] CNRS, LRI, Univ. Paris Sud, Orsay, France [2] Univ. Côte d'Azur, CNRS, Inria, I3S, France [3] Univ. Côte d'Azur, Inria, CNRS, I3S, France

Extended Abstract

The *chromatic number* $\chi(D)$ of a digraph D is the chromatic number of its underlying graph. The *chromatic number* of a class of digraphs $\mathcal{D}$, denoted by $\chi(\mathcal{D})$, is the smallest k such that $\chi(D) \leq k$ for all $D \in \mathcal{D}$, or $+\infty$ if no such k exists. If $\chi(\mathcal{D}) \neq +\infty$, we say that $\mathcal{D}$ *has bounded chromatic number.*

We are interested in the following question : which are the digraph classes $\mathcal{D}$ such that every digraph with sufficiently large chromatic number contains an element of $\mathcal{D}$? Let us denote by Forb(H) (resp. Forb($\mathcal{H}$)) the class of digraphs that do not contain H (resp. any element of $\mathcal{H}$) as a subdigraph. The above question can be restated as follows :

Problem 1. *Which are the classes of digraphs $\mathcal{D}$ such that $\chi(\mathrm{Forb}(\mathcal{D})) < +\infty$?*

An *oriented graph* is an orientation of a (simple) graph. An oriented path (resp., an oriented cycle) is said *directed* if all vertices have in-degree and out-degree at most 1.

Observe that if D is an orientation of a graph G and Forb(D) has bounded chromatic number, then Forb(G) has also bounded chromatic number. A classical result by Erdős implies that G must be a tree. Burr proved that every $(k-1)^2$-chromatic digraph contains every oriented tree of order k and conjectured Burr [3] that it could be further improved to $(2k-2)$-chromatic digraphs.

For special oriented trees T, better bounds on the chromatic number of Forb(T) are known. The most famous one, known as Gallai-Hasse-Roy-Vitaver Theorem [6] states that $\chi(\mathrm{Forb}(P^+(k))) = k$, where $P^+(k)$ is the directed path of length k (a *directed path* is an oriented path in which all arcs are in the same direction).

The chromatic number of the class of digraphs not containing a prescribed oriented path P on n vertices with two blocks (*blocks* are maximal directed subpaths) has been determined by Addario-Berry et al. [1] :

Theorem 2 (Addario-Berry et al. [1])**.** *Let P be an oriented path with two blocks on $n \geq 4$ vertices, then $\chi(\mathrm{Forb}(P)) = n - 1$.*

In this paper, we are interested in the chromatic number of Forb($\mathcal{H}$) when $\mathcal{H}$ is an infinite family of oriented cycles. Let us denote by S-Forb(D) (resp. S-Forb($\mathcal{D}$)) the class of digraphs that contain no subdivision of D (resp. any element of $\mathcal{D}$) as a subdigraph. We are particularly interested in the chromatic number of S-Forb($\mathcal{C}$), where $\mathcal{C}$ is a family of oriented cycles.

Let us denote by $\vec{C}_k$ the directed cycle of length k. For all k, $\chi(\text{S-Forb}(\vec{C}_k)) = +\infty$ because transitive tournaments have no directed cycle. Let us denote by $C(k,\ell)$ the oriented cycle with two blocks, one of length k and the other of length ℓ. Observe that the oriented cycles with two blocks are the subdivisions of $C(1,1)$. As pointed by Gyárfás and Thomassen (see [1]), there are acyclic oriented graphs with arbitrarily large chromatic number and no oriented cycles with two blocks. Therefore $\chi(\text{S-Forb}(C(k,\ell))) = +\infty$. We first generalise this result to every oriented cycle.

Theorem 3. *For any oriented cycle C, $\chi(\text{S-Forb}(C)) = +\infty$.*

In fact, we show the following stronger theorem.

Theorem 4. *For any positive integers b, c, there exists an acyclic digraph D_c with $\chi(D_c) \geq c$ in which all oriented cycles have more than b blocks.*

We need a construction due to Erdős and Lovász [5] of hypergraphs with high girth and large chromatic number.

Theorem 5. *[5, Theorem 1'] For $k, g, c \in \mathbb{N}$, there exists a k-uniform hypergraph with girth larger than g and weak chromatic number larger than c.*

We assume g is being fixed, the following construction allow us to find D_{c+1} from D_c. Let p be the number of proper c-colourings of D_c, and let those colourings be denoted by $col_c^1, ..., col_c^p$. By Theorem 5 there exists a $c \times p$-uniform hypergraph $\mathcal{H}$ with weak chromatic number $> p$ and girth $> g/2$. Let $X = \{x_1, \ldots, x_n\}$ be the ground set of $\mathcal{H}$.

We construct D_{c+1} from n disjoint copies $D_c^1, ..., D_c^n$ of D_c as follows. For each hyperedge $S \in \mathcal{H}$, we do the following :

- We partition S into p sets $S_1, \ldots, S_p$ of cardinality c.
- For each set $S_i = \{x_{k_1}, \ldots, x_{k_c}\}$, we choose vertices $v_{k_1} \in D_c^{k_1}, \ldots, v_{k_c} \in D_c^{k_c}$ such that $col_c^i(v_{k_1}) = 1, \ldots, col_c^i(v_{k_c}) = c$, and add a new vertex $w_{S,i}$ with $v_{k_1}, \ldots, v_{k_c}$ as in-neighbours.

On the other hand, considering strongly connected (strong for short) digraphs may lead to dramatically different result. An example is provided by the following celebrated result due to Bondy [2], which can be rephrased as follows when denoting the class of strong digraphs by $\mathcal{S}$.

Theorem 6 (Bondy [2])**.** $\chi(\text{S-Forb}(\vec{C}_k) \cap \mathcal{S}) = k - 1$.

Inspired by this theorem, Addario-Berry et al. [1] posed the following problem.

Problem 7. *Let k and ℓ be two positive integers then $\chi(\text{S-Forb}(C(k,\ell) \cap \mathcal{S})) < k + l$.*

We give evidence for this problem by showing the following weaker statement.

Theorem 8. *Let k and ℓ be two positive integers such that $k \geq \max\{\ell, 3\}$, and let D be a digraph in $\text{S-Forb}(C(k,\ell)) \cap \mathcal{S}$. Then, $\chi(D) \leq (k+\ell-2)(k+\ell-3)(2\ell+2)(k+\ell+1)$.*

We need the following lemma.

The *union* of two digraphs D_1 and D_2 is the digraph $D_1 \cup D_2$ with vertex set $V(D_1) \cup V(D_2)$ and arc set $A(D_1) \cup A(D_2)$.

Lemma 9. *Let D_1 and D_2 be two digraphs. $\chi(D_1 \cup D_2) \leq \chi(D_1) \times \chi(D_2)$.*

A consequence of the previous lemma is that, if we partition the arc set of D into set A_1 $\cdots$ A_k, then bounding the chromatic number of all digraphs induced by the A_i implies that D has bounded chromatic number.

Proof. Let D be a strong digraph without any copy of $C(k,\ell)$, we exhibit a colouring of D using a bounded number of colours. The proof heavily relies on the technique of *levelling*. Let u be a vertex of D. The *level* of a vertex x, noted $\text{lvl}(x)$ is the length of the shortest dipath from u to x. $L(i)$ is the set of vertices at level i.

Since D is strongly connected, it has an out-generator u. Let T be a BFS-tree with root u. We define the following sets of arcs.

$$\begin{aligned}
A_0 &= \{xy \in A(D) \mid \text{lvl}(x) = \text{lvl}(y)\}; \\
A_1 &= \{xy \in A(D) \mid 0 < |\text{lvl}(x) - \text{lvl}(y)| < k + \ell - 3; \\
A' &= \{xy \in A(D) \mid \text{lvl}(x) - \text{lvl}(y) \geq k + \ell - 3\}.
\end{aligned}$$

Since $k + \ell - 3 > 0$ and there is no arc xy with $\text{lvl}(y) > \text{lvl}(x) + 1$, (A_0, A_1, A') is a partition of $A(D)$. Observe moreover that $A(T) \subseteq A_1$. We further partition A' into two sets A_2 and A_3, where $A_2 = \{xy \in A' \mid y \text{ is an ancestor of } x \text{ in } T\}$ and $A_3 = A' \setminus A_2$. Then (A_0, A_1, A_2, A_3) is a partition of $A(D)$. Let $D_j = (V(D), A_j)$ for all $j \in \{0, 1, 2, 3\}$.

Claim 10. $\chi(D_0) \leq k + \ell - 2$.

Proof. Observe that D_0 is the disjoint union of the $D[L_i]$ where $L_i = \{v \mid \mathrm{dist}_D(u,v) = i\}$. Therefore it suffices to prove that $\chi(D[L_i]) \leq k + \ell - 2$ for all non-negative integer i.

$L_0 = \{u\}$ so the result holds trivially for $i = 0$.

Assume now $i \geq 1$. Suppose for a contradiction $\chi(D[L_i]) \geq k + \ell - 1$. Since $k \geq 3$, by Theorem 2, $D[L_i]$ contains a copy Q of $P^+(k-1, \ell-1)$, the path on two blocks of length $k-1$ and $\ell-1$ with one vertex of indegree 2. Let v_1 and v_2 be the initial and terminal vertices of Q, and let x be the least common ancestor of v_1 and v_2. By definition, for $j \in \{1,2\}$, there exists a dipath P_j from x to v_j in T. By definition of least common ancestor, $V(P_1) \cap V(P_2) = \{x\}$, $V(P_j) \cap L_i = \{v_j\}$, $j = 1, 2$, and both P_1 and P_2 have length at least 1. Consequently, $P_1 \cup P_2 \cup Q$ is a subdivision of $C(k, \ell)$, a contradiction. ☐

Claim 11. $\chi(D_1) \leq k + \ell - 3$.

Proof. Let ϕ_1 be the colouring of D_1 defined by $\phi_1(x) = \mathrm{lvl}(x) \pmod{k+\ell-3}$. By definition of D_1, this is clearly a proper colouring of D_1. ☐

The following two claims are more complicated, we refer the reader to [4] for the complete proofs.

Claim 12. $\chi(D_2) \leq 2\ell + 2$.

Claim 13. $\chi(D_3) \leq k + \ell + 1$.

Claims 10, 11, 12, and 13, together with Lemma 9 yield the result.

☐

More generally, one may wonder what happens for other oriented cycles. Our next result generalises Theorem 8 for $\hat{C}_4$ the cycle with 4 blocks.

Theorem 14. *Let D be a digraph in* S-Forb$(\hat{C}_4)$*. If D admits an out-generator, then* $\chi(D) \leq 24$.

Proof. The general idea is the same as in the proof of Theorem 8.

Suppose that D admits an out-generator u and let T be an BFS-tree with root u. We partition $A(D)$ into three sets according to the levels of u.

$$\begin{aligned}
A_0 &= \{(x,y) \in A(D) \mid \mathrm{lvl}(x) = \mathrm{lvl}(y)\};\\
A_1 &= \{(x,y) \in A(D) \mid |\mathrm{lvl}(x) - \mathrm{lvl}(y)| = 1\};\\
A_2 &= \{(x,y) \in A(D) \mid \mathrm{lvl}(y) \leq \mathrm{lvl}(x) - 2\}.
\end{aligned}$$

For $i = 0, 1, 2$, let $D_i = (V(D), A_i)$.

Claim 15. $\chi(D_0) \leq 3$.

Proof. Suppose for a contradiction that $\chi(D) \geq 4$. By Theorem 2, it contains a $P^-(1,1)$ (y_1, y, y_2), that is (y, y_1) and (y, y_2) are in $A(D_0)$. Let x be the least common ancestor of y_1 and y_2 in T. The union of $T[x, y_1]$, (y, y_1), (y, y_2), and $T[x, y_2]$ is a subdivision of $\hat{C}_4$, a contradiction. ☐

Claim 16. $\chi(D_1) \leq 2$.

Proof. Since the arc are between consecutive levels, then the colouring ϕ_1 defined by $\phi_1(x) = lvl(x) \mod 2$ is a proper 2-colouring of D_1. ☐

Let $y \in V_i$ we denote by $N'(y)$ the out-degree of y in $\bigcup_{0 \leq j \leq i-1} V_j$. Let $D' = (V, A')$ with $A' = \cup_{x \in V}\{(x,y), y \in N'(x)\}$ and $D_x = (V, A_x)$ where A_x is the set of arc inside the level and from V_i to V_{i+1} for all i. Note that $A = A' \cup A_x$ and

Claim 17. $\chi(D_2) \leq 4$.

Proof. We refer to [4] for the proof of this statement. $\square$

Claims 15, 16, 17, and Lemma 9 implies $\chi(D) \leq 24$. $\square$

References

[1] L. Addario-Berry, F. Havet, and S. Thomassé. Paths with two blocks in n-chromatic digraphs. *Journal of Combinatorial Theory, Series B*, 97 (4): 620–626, 2007.

[2] J. A. Bondy, Disconnected orientations and a conjecture of Las Vergnas, *J. London Math. Soc. (2)*, **14** (2) (1976), 277–282.

[3] S. A. Burr. Subtrees of directed graphs and hypergraphs. In *Proceedings of the 11th Southeastern Conference on Combinatorics, Graph theory and Computing*, pages 227–239, Boca Raton - FL, 1980. Florida Atlantic University.

[4] N. Cohen, F. Havet, W. Lochet and N.Nisse, Subdivisions of oriented cycles in digraphs with large chromatic number. *arXiv preprint arXiv:1605.07762*, 2016.

[5] P. Erdős and L. Lovász. Problems and results on 3-chromatic hypergraphs and some related questions. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II*, pages 609–627. Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, Amsterdam, 1975.

[6] T. Gallai. On directed paths and circuits. In *Theory of Graphs (Proc. Colloq. Titany, 1966)*, pages 115–118. Academic Press, New York, 1968.